Exercises
Spontaneous Symmetry Breaking and Field Theory 1

Sheet 1

1/1 Non-interacting spins

Calculate for a system of N spins $\widehat{\mathbf{S}}_i$ with spin quantum number S, which do not mutually interact with each other but which interact with an external field $\mathbf{H}$ according to

$$\widehat{H} = -\mathbf{H} \cdot \sum_{i=1}^{N} \widehat{\mathbf{S}}_i \ , \tag{1}$$

the partition function, the magnetic free enthalpy, the entropy, the magnetic moment and the susceptibility.

1/2 One-dimensional classical Ising chain

Show that the one-dimensional classical Ising model with N spins, $S_i = \pm 1$ for all sites i, does not have a magnetic moment at non-zero temperature for zero external field $\mathbf{H} = 0$. To do this, calculate the thermodynamic expectation value

$$\overline{S_k S_{k+r}} = \frac{1}{Z_N} \sum_n S_{k,n} S_{k+r,n} \exp\left(\frac{-E_n}{k_B T}\right) , \tag{2}$$

where Z_N is the canonical partition function and where the sum runs over all conceivable microconfigurations $\{S_{i,n}\}$ with energies

$$E_n = -\sum_{i=1}^{N-1} J_i \ S_{i,n} S_{i+1,n} \ , \qquad J_i = J > 0. \tag{3}$$

<u>Hint:</u> Assume first that all J_i are different for different i, and show

$$\overline{S_k S_{k+r}} \ Z_N = \frac{\partial}{\partial \widetilde{J}_k} \frac{\partial}{\partial \widetilde{J}_{k+1}} \cdots \frac{\partial}{\partial \widetilde{J}_{k+r-1}} Z_N \tag{4}$$

with $\widetilde{J}_i = J_i/(k_B T)$. Then add to the system with N spins a further spin and express Z_{N+1} via Z_N. Calculate with the so obtained recursion formula the partition function Z_N, and set $J_i = J$ for all i at the end.

1/3 Ising ring

Calculate the partition function and the magnetic moment for the one-dimensional classical nearest-neighbour Ising ring (chain with periodic boundary conditions) and the interaction operator

$$\widehat{H} = -J \sum_{\substack{\text{nearest-}\\ \text{neighbour}\\ \text{pairs } ij}} S_i S_j - H \sum_i S_i \ , \qquad S_i = \pm 1 \tag{5}$$

in the external field H.
<u>Hint:</u> Show that the partition function Z_N for N spins may be written as

$$Z_N(T,H) = \mathrm{Tr}\left(\left(\underline{\underline{P}} \right)^N \right) \tag{6}$$

with the transfer matrix

$$P_{12} = P_{21} = \exp\left(\frac{-J}{k_B T} \right) \tag{7}$$

$$P_{11} = \exp\left(\frac{J+H}{k_B T} \right) \tag{8}$$

$$P_{22} = \exp\left(\frac{J-H}{k_B T} \right) . \tag{9}$$