Exercises
Spontaneous Symmetry Breaking and Field Theory

Sheet 3

3/1 Alternative approach to molecular-field results

In the lectures the molecular-field theory has been introduced by replacing the Heisenberg exchange Hamiltonian for spins with nearest neighbour interaction by the molecular-field Hamiltonian

$$H^{\mathrm{MF}} = -2qJ\overline{S^z}\sum_i S_i^z \ , \tag{1}$$

where q is the coordination number, J the exchange integral and $\overline{S^z}$ the thermal average of S_i^z (magnetic field $\mathbf{H} = (0,0,H)^T$). There is an alternative way to derive the molecular-field results, for which the analogy to the Landau-Ginzburg theory can be very well visualized. Thereby the partition function of N Ising spins $S_i = \pm\frac{1}{2}$ is approximated by ($S = \sum_i S_i$)

$$Y(T,H) = \sum_{S=-\frac{N}{2}}^{+\frac{N}{2}} \sum_{\{S_i\}\to S} \exp\left[\frac{1}{kT}\left(H + \widetilde{H}^{\mathrm{MF}}(S)\right)S\right] = \sum_{S=-\frac{N}{2}}^{+\frac{N}{2}} g_S \exp\left[\frac{1}{kT}\left(H + \widetilde{H}^{\mathrm{MF}}(S)\right)S\right],$$

with $\quad \widetilde{H}^{\mathrm{MF}}(S) = 2qJ\dfrac{S}{N}.$ (2)

This means that the Heisenberg Hamiltonian

$$H = -\frac{1}{2}J{\sum_{i,j}}' S_iS_j \tag{3}$$

where the sum runs over all nearest-neighbour pairs (i,j) is replaced for all configurations $\{S_i\}$ leading to the same S by $-\frac{1}{2}qJS^2/N$. Furthermore, g_S is the number of configurations leading to the same S. Calculate g_S and approximate the obtained expression by use of Stirling's formula

$$\ln N! \approx N\ln N - N. \tag{4}$$

Look for the value $\overline{S}$ of S giving the largest contribution to $Y(T,H)$, and show that this yields the molecular-field equation for $m = \overline{S}/(N/2)$,

$$m = \tanh\left(\frac{H + 2qJm}{2kT}\right). \tag{5}$$

Calculate the magnetic free enthalpy

$$G_{\mathrm{magn}} = -kT\ln\left(Y(T,H)\right) \tag{6}$$

by taking into account in $Y(T,H)$ only the contribution of $\overline{S}$, and show that for $T \approx T_C$ (m small) an equation analogous to the Landau-Ginzburg form of G_{magn} is found.

3/2 Ginzburg criterion

Calculate by the Landau-Ginzburg theory for $\mathbf{H} = 0$ the magnetic specific heat $c(T, \mathbf{H} = 0)$ and show that this quantity jumps at T_C.

Calculate for $\mathbf{H} = 0$ again the magnetic specific heat $\widetilde{c}(T, \mathbf{H} = 0)$ by the Gauß model. To do this, first calculate $H_{\text{eff}}[\boldsymbol{\sigma}]$ for this model by using a Fourier series for $\boldsymbol{\sigma}(\mathbf{r})$,

$$\boldsymbol{\sigma}(\mathbf{r}) = L^{-\frac{d}{2}} \sum_{\mathbf{k}, |\mathbf{k}| < \Lambda} \boldsymbol{\sigma}_{\mathbf{k}} \exp(i\mathbf{kr}), \tag{7}$$

where L is the linear extension of the d-dimensional system and Λ is a cut-off wave vector which expresses the fact that $\boldsymbol{\sigma}(\mathbf{r})$ is a block-spin variable and not an atomistic variable. This yields

$$H_{\text{eff}}[\boldsymbol{\sigma}] = \sum_{\mathbf{k}, |\mathbf{k}| < \Lambda} \frac{1}{2}(a_2 + ck^2) \left|\boldsymbol{\sigma}_{\mathbf{k}}\right|^2. \tag{8}$$

Calculate the partition function

$$Y(T, \mathbf{H} = 0) = \prod_{\mathbf{k}, |\mathbf{k}| < \Lambda} 2 \int_0^\infty d|\boldsymbol{\sigma}_{\mathbf{k}}| \exp\left(-\frac{H_{\text{eff}}[\boldsymbol{\sigma}]}{kT}\right), \tag{9}$$

the corresponding magnetic free enthalpy and from that the term which dominates $\widetilde{c}(T, \mathbf{H} = 0)$ at $T \approx T_C$. Thereby use in this last step the continuization

$$\sum_{\mathbf{k}} \to \frac{L^d}{(2\pi)^d} \int d^d\mathbf{k}. \tag{10}$$

Discuss the Ginzburg-criterion introduced in the lectures by considering $\widetilde{c}/\Delta c$.