Exercises
Spontaneous Symmetry Breaking and Field Theory 1

Sheet 5

5/1 Real-space renormalization for the Ising model on a triangular lattice

We consider a system of Ising spins with nearest-neighbour coupling J ($K = J/(k_B T)$) on a triangular lattice for zero magnetic field H (scaling factor $b = \sqrt{3}$). By the renormalization transformation new couplings L and M between second-nearest neighbours and third-nearest neighbours are generated. The transformation yields

$$K' = 2a_1^2 K + 4(p+q)a_1^2 K^2 + 3a_1^2 L + 2a_1^2 M \tag{1}$$
$$L' = (p+7q)a_1^2 K^2 + a_1^2 M \tag{2}$$
$$M' = 4qa_1^2 K^2 \tag{3}$$

with

$$p = 1 - a_1^2; \quad q = a_2 - a_1^2; \quad a_i = n_i/z; \tag{4}$$
$$n_1 = \exp(3K) + \exp(-K); \quad n_2 = \exp(3K) - \exp(-K); \tag{5}$$
$$z = \exp(3K) + 3\exp(-K). \tag{6}$$

There is one non-trivial fixed point of this renormalization transformation:

$$K^* = 0.2789; \quad L^* = -0.0143; \quad M^* = -0.0152. \tag{7}$$

a) Calculate the transformation matrix $\underline{\underline{R}}^L$ for the linearized renormalization transformation

$$\boldsymbol{\mu}' = \begin{pmatrix} K' \\ L' \\ M' \end{pmatrix} = \boldsymbol{\mu}^* + \underline{\underline{R}}^L \delta\boldsymbol{\mu} = \begin{pmatrix} K^* \\ L^* \\ M^* \end{pmatrix} + \underline{\underline{R}}^L \begin{pmatrix} K - K^* \\ L - L^* \\ M - M^* \end{pmatrix}. \tag{8}$$

Calculate the eigenvalues. What is the relevant eigenvalue (which we call λ_1)? Calculate the exponent y_t which describes the renormalization of $t = (T - T_C)/T_C$ upon rescaling, i.e., $t' = b^{y_t} t$, and the exponent ν of the correlation length.

b) The unrenormalized system is at the point $\boldsymbol{\mu} = \boldsymbol{\mu}^* + \delta\boldsymbol{\mu}$ in the parameter space. The quantity $\delta\boldsymbol{\mu}$ can be represented as

$$\delta\boldsymbol{\mu} = \sum_\nu c_\nu \boldsymbol{e}_\nu^r, \tag{9}$$

where the $\boldsymbol{e}_\nu^r$ are the right-hand-side eigenvectors of $\underline{\underline{R}}^L$. For arbitrary temperatures T the repeated application of the renormalization transformation drives $\delta\boldsymbol{\mu}$ away from $\boldsymbol{\mu}^*$. In contrast, for $T = T_C$ $\delta\boldsymbol{\mu}$ is driven to zero, because then the unrenormalized system is on the critical surface $\boldsymbol{\mu}_c = \boldsymbol{\mu}^* + \delta\boldsymbol{\mu}_c$ with $\delta\boldsymbol{\mu}_c = \sum_\nu c_\nu^c \boldsymbol{e}_\nu^r$. This is only possible if the coefficient c_1^c of the relevant eigenvector $\boldsymbol{e}_1^r$ is zero. Calculate c_ν^c for given (unrenormalized) $\delta\boldsymbol{\mu}_c$ by multiplying $\delta\boldsymbol{\mu}_c$ from the left with the left-hand-side eigenvector $\boldsymbol{e}_\nu^l$ of $\underline{\underline{R}}^L$ and by using the orthogonality relation

$$\boldsymbol{e}_\nu^l \cdot \boldsymbol{e}_{\nu'}^r = N_\nu \delta_{\nu\nu'}. \tag{10}$$

Doing this for $\nu = 1$ yields an equation for $T_C = J/(k_B K_c)$.